Asymptotics & Disjoint Sets

Exam-Level 05

Announcements

Sunday	Monday	Tuesday	Wednesday	Thursday	Friday	Saturday
		2/20 Lab 4 Due Project 1C Due				
	2/26 Lab 5 Due Homework 2 Due					

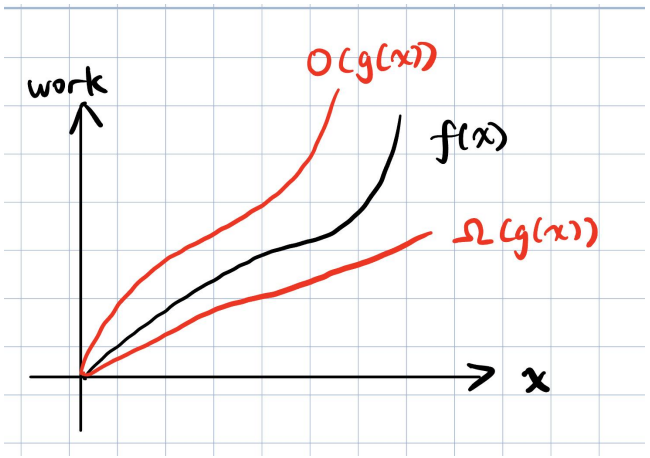
Content Review

Asymptotics

Asymptotics allow us to evaluate the performance of programs using math.

We ignore all constants and only care about the total work done when the input is very large.

Big O - If a function f(x) has big O in g(x), it grows at most as fast as g(x).
Big Ω - f(x) grows at least as fast as g(x),
Big Θ - When a function is both O(g(x)) and Ω(g(x)), it is Θ(g(x))



Common Orders of Growth

- $O(1) < O(\log n) < O(n) < O(n \log n) < O(n^2) < O(c^n)$
- Alternatively: constant < logarithmic < linear < nlogn < quadratic < exponential
- [Desmos example here]
    - Constants don't matter in the long run!
- Fun sums:

$$1 + 2 + 3 + \ldots + N = \Theta(N^2)$$
$$1 + 2 + 4 + 8 + \ldots + N = \Theta(N)$$
$$1 + 2 + 4 + 8 + \ldots + 2^N = \theta(2^N)$$

Tightest Bound?

- Sometimes it's easier to bound the runtime than to calculate the runtime.
- When you bound, always provide the tightest bound
  - i.e, the bound that provides the most information about the runtime.
- Ex. Given $f(n) = 2n + 5$, we could say that $f(n) \in O(n^n)$, but that doesn't tell us very much
  - a lot of functions are upper bounded by $O(n^n)$ (grows really fast!)
  - A better, tighter bound would be $f(n) \in \Theta(n)$

Best vs. Worst Case

- In best-worst case analysis, we still assume the input is very large.
- Therefore, you cannot make assumptions such as N == 1 or N <= 10 in these analyses.
- **The best case is not when the input is 1.**
- **The best case is not when the input is 1.**
- **Seriously, the best case is not when the input is 1.**

Best vs. Worst Case

- Represented with tight bound Θ because they should be consistent (always run in the same time)
- Look out for branching statements, loop conditions, breaks
- Check: What is the best/worst case runtime of the function below?
- **Remember: The best case is not when the input is 1.**

```
public static void example(int N) {
     while (N > 0) {
          if (func(N)) {
               break;
          }
          N -= 1;
     }
}
```

Best case: Θ(1), where N = some int for which func(N) is immediately true

I'm not assuming N is 1 or N is small. func(N) could be return whether N is an even number, and when N is very large but even number this function runs in constant time

Worst case: Θ(N), where N = some int for which func(N), func(N - 1), …, func(1) are all false

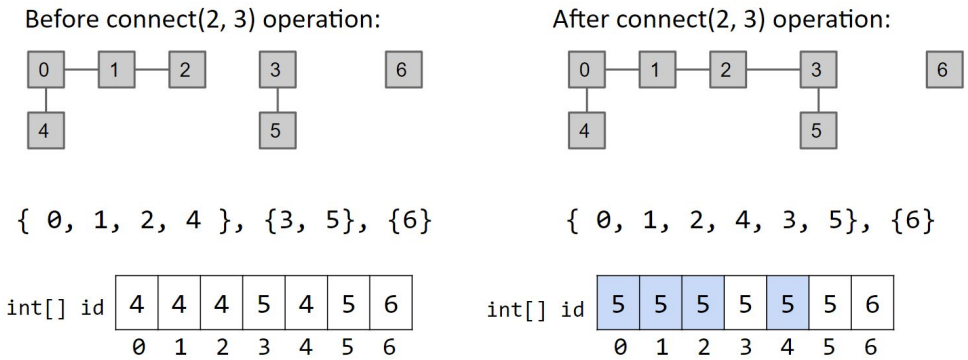
Best vs. Worst Case

- Best/worst case vs. lower/upper bound analogy: how much does it cost to eat at a restaurant?
  - Best/worst-case: “the cheapest thing on the menu is $5 and the most expensive is $50”
  - Lower/upper bound: “every item is at least $5 and at most $50” (credit: Alex Schedel)
- Which one is more informative?
  - The first one: the best/worst-case are the tightest lower/upper-bounds you can give.

Disjoint Sets, also known as Union Find

```
public interface DisjointSet {
     void connect (x, y); // Connects nodes x and y (you may also see union)
     boolean isConnected(x, y); // Returns true if x and y are connected
}
```

QuickFind uses an array of integers to track which set each element belongs to.



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public interface DisjointSet {
     void connect (x, y); // Connects nodes x and y (you may also see union)
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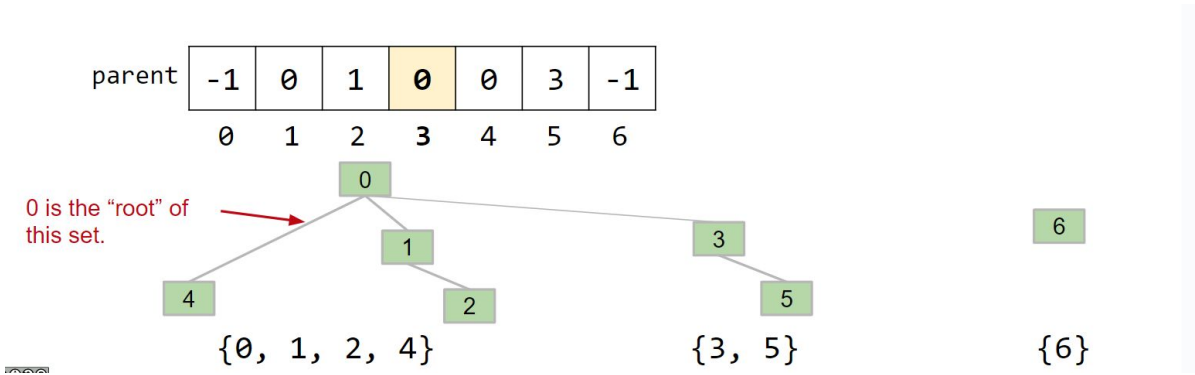
}
```

QuickUnion stores the parent of each node rather than the set to which it belongs and merges sets by setting the parent of one root to the other.



Disjoint Sets, also known as Union Find

```
public interface DisjointSet {
     void connect (x, y); // Connects nodes x and y (you may also see union)
     boolean isConnected(x, y); // Returns true if x and y are connected
}
```

WeightedQuickUnion does the same as QuickUnion except it decides which set is merged into which by size (merge smaller into larger), reducing stringiness.

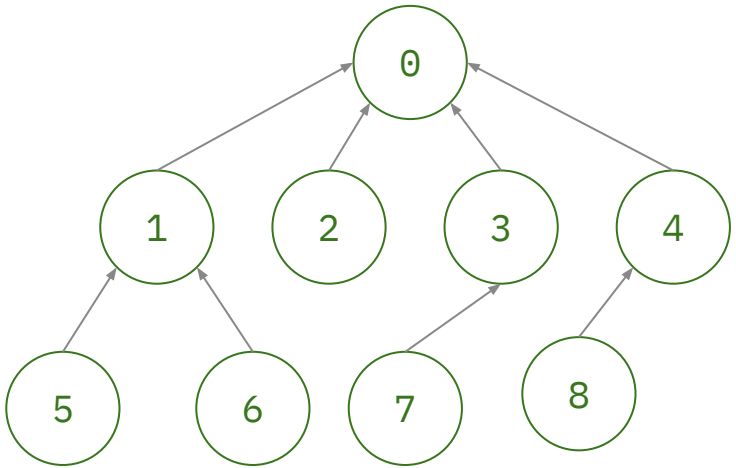
WeightedQuickUnion with Path Compression sets the parent of each node to the set's root whenever find() is called on it.

Disjoint Sets Representation

- We can use a single array to represent our disjoint set when implementing `connect()` optimally (ie. `WeightedQuickUnion`)

- `arr[i]` contains the parent of element i in the set; the index of a root contains - (`# elements in set rooted at that index`)

```
[-9, 0, 0, 0, 0, 1, 1, 3, 4]
```



Disjoint Sets Asymptotics

```
public interface DisjointSet {
     void connect (x, y); // Connects nodes x and y (you may also see union)
     boolean isConnected(x, y); // Returns true if x and y are connected
}
```

Implementation	Constructor	connect()	isConnected()
QuickUnion	Θ(N)	O(N)	O(N)
QuickFind	Θ(N)	O(N)	O(1)
Weighted Quick Union	Θ(N)	O(log N)	O(log N)
WQU with Path Compression	Θ(N)	O(log N) Θ(1)-ish amortized	O(log N) Θ(1)-ish amortized

* we don't really talk about QU/QF in application, more to show the asymptotic motivation for WQU

Worksheet

1A Asymptotics Introduction

```
private void f1(int N) {
    for (int i = 1; i < N; i++) {
        for (int j = 1; j < i; j++) {
            System.out.println("shreyas 1.0");
        }
    }
}
```


1A Asymptotics Introduction

```
private void f1(int N) {
    for (int i = 1; i < N; i++) {
        for (int j = 1; j < i; j++) {
            System.out.println("shreyas 1.0");
        }
    }
}
```

i	1	2	…	N-2	N-1
Work per i	0	1	…	N-3	N-2

$1 + 2 + 3 + 4 + \ldots N-1 = \Theta(N^2)$

1B Asymptotics Introduction

```
private void f2(int N) {
    for (int i = 1; i < N; i *= 2) {
        for (int j = 1; j < i; j++) {
            System.out.println("shreyas 2.0");
        }
    }
}
```


1B Asymptotics Introduction

```
private void f2(int N) {
    for (int i = 1; i < N; i *= 2) {
        for (int j = 1; j < i; j++) {
            System.out.println("shreyas 2.0");
        }
    }
}
```

i	1	2	4	…	N/4	N/2
Work per i	0	1	3	…	N/4 - 1	N/2 - 1

$1 + 2 + 4 + 8 + \ldots N/4 + N/2 = \Theta(N)$

2 Disjoint Sets

For each of the arrays below, write whether this could be the array representation of a weighted quick union with path compression.

	i:	0	1	2	3	4	5	6	7	8	9
A.	a[i]:	1	2	3	0	1	1	1	4	4	5
B.	a[i]:	9	0	0	0	0	0	9	9	9	-10
C.	a[i]:	1	2	3	4	5	6	7	8	9	-10
D.	a[i]:	-10	0	0	0	0	1	1	1	6	2
E.	a[i]:	-10	0	0	0	0	1	1	1	6	8
F.	a[i]:	-7	0	0	1	1	3	3	-3	7	7

2 Disjoint Sets

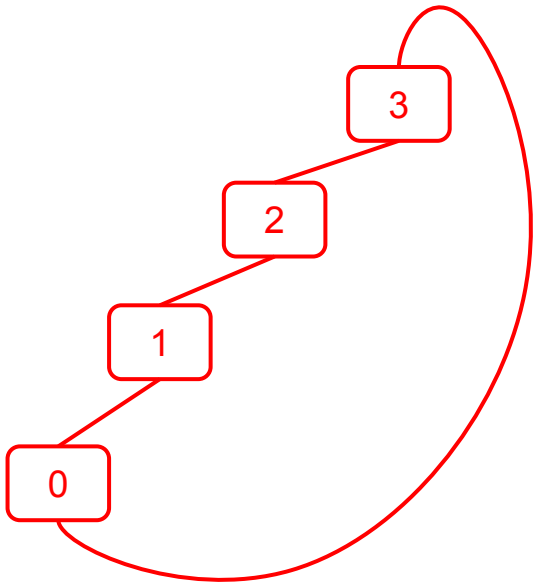
There are three criteria here that invalidates a representation:

- There is a cycle in the parent-link.
- For each parent-child link, the tree rooted at the parent is smaller than the tree rooted at the child before the link (you would have merged the other way around).
- The height of the tree is greater than log2(n), where n is the number of elements. Height is defined as the number of edges counting from a “root” to “leaf”.

2 Disjoint Sets

For each of the arrays below, write whether this could be the array representation of a weighted quick union with path compression.

	i:	0	1	2	3	4	5	6	7	8	9
A.	a[i]:	1	2	3	0	1	1	1	4	4	5
B.	a[i]:	9	0	0	0	0	0	9	9	9	-10
C.	a[i]:	1	2	3	4	5	6	7	8	9	-10
D.	a[i]:	-10	0	0	0	0	1	1	1	6	2
E.	a[i]:	-10	0	0	0	0	1	1	1	6	8
F.	a[i]:	-7	0	0	1	1	3	3	-3	7	7

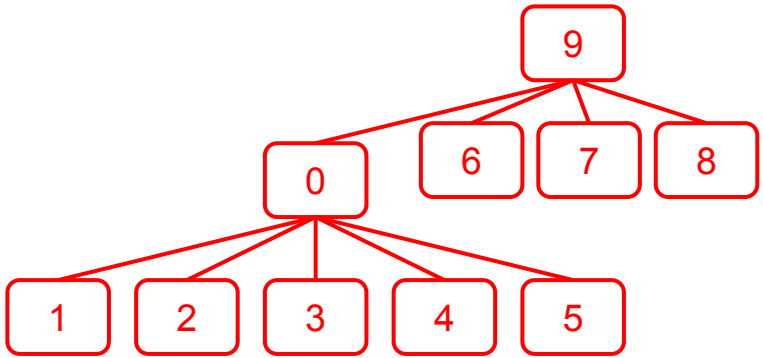


Impossible: cycle 0-1, 1-2, 2-3, and 3-0 in the parent-link representation.

2 Disjoint Sets

For each of the arrays below, write whether this could be the array representation of a weighted quick union with path compression.

	i:	0	1	2	3	4	5	6	7	8	9
A.	a[i]:	1	2	3	0	1	1	1	4	4	5
B.	a[i]:	9	0	0	0	0	0	9	9	9	-10
C.	a[i]:	1	2	3	4	5	6	7	8	9	-10
D.	a[i]:	-10	0	0	0	0	1	1	1	6	2
E.	a[i]:	-10	0	0	0	0	1	1	1	6	8
F.	a[i]:	-7	0	0	1	1	3	3	-3	7	7



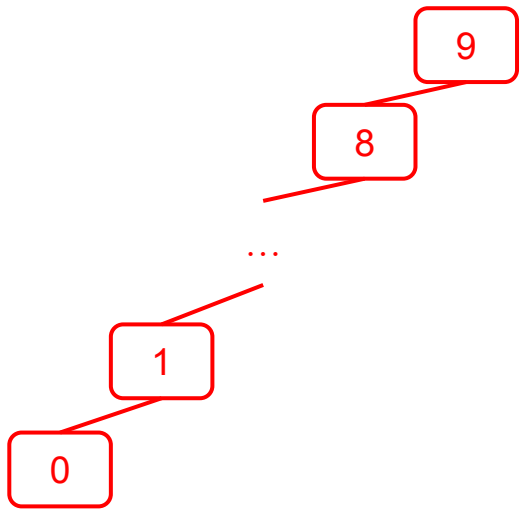
Impossible: 1, 2, 3, 4, and 5 must link to 0 when 0 is a root; 0 would not link to 9 because 0 is the root of the larger tree.

2 Disjoint Sets

For each of the arrays below, write whether this could be the array representation of a weighted quick union with path compression.

	i:	0	1	2	3	4	5	6	7	8	9
A.	a[i]:	1	2	3	0	1	1	1	4	4	5
B.	a[i]:	9	0	0	0	0	0	9	9	9	-10
C.	a[i]:	1	2	3	4	5	6	7	8	9	-10
D.	a[i]:	-10	0	0	0	0	1	1	1	6	2
E.	a[i]:	-10	0	0	0	0	1	1	1	6	8
F.	a[i]:	-7	0	0	1	1	3	3	-3	7	7

Impossible: tree rooted at 9 has height 9 > lg 10.



2 Disjoint Sets

For each of the arrays below, write whether this could be the array representation of a weighted quick union with path compression.

	i:	0	1	2	3	4	5	6	7	8	9
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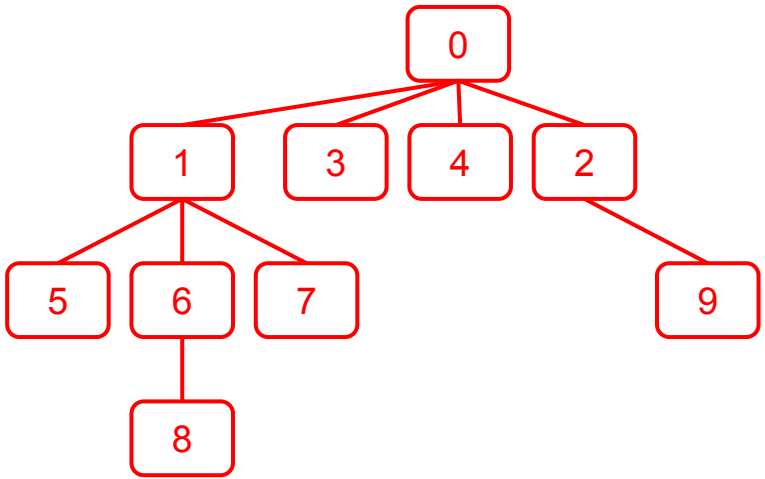
C.	a[i]:	1	2	3	4	5	6	7	8	9	-10
D.	a[i]:	-10	0	0	0	0	1	1	1	6	2
E.	a[i]:	-10	0	0	0	0	1	1	1	6	8
F.	a[i]:	-7	0	0	1	1	3	3	-3	7	7

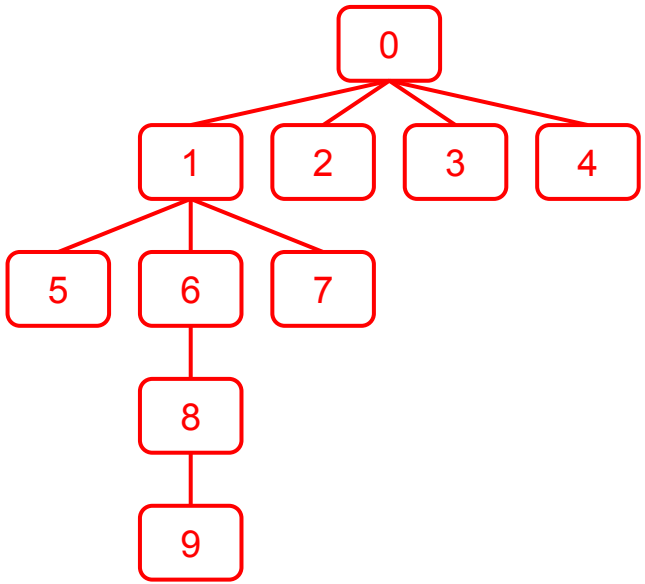


Possible: 8-6, 7-1, 6-1, 5-1, 9-2, 3-0, 4-0, 2-0, 1-0.

2 Disjoint Sets

For each of the arrays below, write whether this could be the array representation of a weighted quick union with path compression.

	i:	0	1	2	3	4	5	6	7	8	9
A.	a[i]:	1	2	3	0	1	1	1	4	4	5
B.	a[i]:	9	0	0	0	0	0	9	9	9	-10
C.	a[i]:	1	2	3	4	5	6	7	8	9	-10
D.	a[i]:	-10	0	0	0	0	1	1	1	6	2
E.	a[i]:	-10	0	0	0	0	1	1	1	6	8
F.	a[i]:	-7	0	0	1	1	3	3	-3	7	7

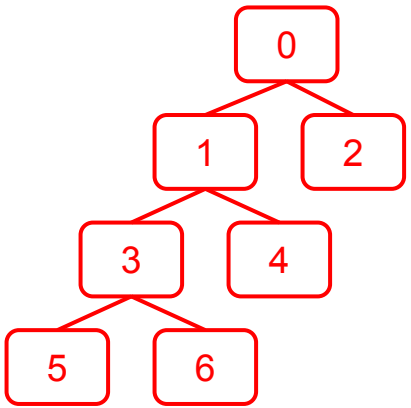


Impossible: tree rooted at 0 has height 4 > lg 10.

2 Disjoint Sets

For each of the arrays below, write whether this could be the array representation of a weighted quick union with path compression.

	i:	0	1	2	3	4	5	6	7	8	9
A.	a[i]:	1	2	3	0	1	1	1	4	4	5
B.	a[i]:	9	0	0	0	0	0	9	9	9	-10
C.	a[i]:	1	2	3	4	5	6	7	8	9	-10
D.	a[i]:	-10	0	0	0	0	1	1	1	6	2
E.	a[i]:	-10	0	0	0	0	1	1	1	6	8
F.	a[i]:	-7	0	0	1	1	3	3	-3	7	7



Impossible: tree rooted at 0 has height 3 > lg 7.

3A Asymptotics of Weighted Quick Unions

Suppose we have a Weighted Quick Union (WQU) without path compression with N elements.

1. What is the runtime, in big Ω and big *O*, of isConnected?
2. What is the runtime, in big Ω and big *O*, of connect?

3A Asymptotics of Weighted Quick Unions

Suppose we have a Weighted Quick Union (WQU) without path compression with N elements.

1. What is the runtime, in big Ω and big *O*, of isConnected? **Ω(1), O(log(N))**

2. What is the runtime, in big Ω and big *O*, of connect? **Ω(1), O(log(N))**

3B Asymptotics of Weighted Quick Unions

```
void addToWQU(int[] elements) {
    int[][] pairs = pairs(elements); // constant, returns pairs in random order
    for (int[] pair: pairs) {
        if (size() == elements.length) { // constant
            return;
        }
        connect(pair[0], pair[1]);
    }
}
```


3B Asymptotics of Weighted Quick Unions

```
void addToWQU(int[] elements) {
    int[][] pairs = pairs(elements);
    for (int[] pair: pairs) {
        if (size() == elements.length) {
            return;
        }
        connect(pair[0], pair[1]);
    }
}
```

Best case: $\Omega(N)$
Connect all elements in first N calls, with O(1) connect

Worst case: $O(N^2 \log(N))$
One isolated element until last N calls → N^2
- N iterations, log N connect

3C Asymptotics of Weighted Quick Unions

Define a **matching size connection** as connecting two trees, i.e. components in a WQU, together of matching size. What is the minimum and maximum number of matching size connections that can occur after executing addToWQU?

Assume N, i.e. elements.length, is a power of two.

3C Asymptotics of Weighted Quick Unions

Define a **matching size connection** as connecting two trees, i.e. components in a WQU, together of matching size. What is the minimum and maximum number of matching size connections that can occur after executing addToWQU?

Assume N, i.e. elements.length, is a power of two.

Min: 1
Only one matching-size connection at beginning

Max: N - 1
Connect equal-size tree pairs until all are connected